Tensor Product and Acyclic Edge Colouring

by

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Declaration

I hereby declare that

i) the thesis comprises of my original work towards the degree of Master of Technology in Information and Communication Technology at Dhirubhai Ambani Institute of Information and Communication Technology and has not been submitted elsewhere for a degree,

ii) due acknowledgment has been made in the text to all the reference material used.

RUTVI SHIHORA

Certificate

This is to certify that the thesis work entitled TENSOR PRODUCT AND ACYCLIC EDGE COLOURING has been carried out by RUTVI SHIHORA for the degree of Master of Technology in Information and Communication Technology at *Dhirubhai Ambani Institute of Information and Communication Technology* under my supervision.

Prof. Rahul Muthu
Thesis Supervisor

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Contents

Abstract

The assignment of colours to the edges of graph G such that no two adjacent edges get the same colour and there is no 2-coloured cycle in G is known as Acyclic Edge Colouring. The minimum number of colours needed to acyclically edge colour the graph is known as acyclic chromatic index and is denoted by $a'(G)$. The difference between $a'(G)$ and $\triangle$(G) is known as the *gap* of the graph. Determining the value of $a'(G)$ either algorithmically or theoretically has been a very difficult problem. It belongs to the class of NP-complete problems. The value of $a'(G)$ has not yet been determined even for the highly structured class of graphs like complete graphs.Determining the exact values of $a'(G)$ even for very special classes of graphs is still open.

Tensor Product, also known as Kronecker Product is a special type of graph product. Any graph that can be represented as a tensor product has the same number of irreducible factors, even though the factor graphs may be different for different factorizations. There exists an algorithm that recognizes tensor product graphs in polynomial time and finds a factorization for the same.

This thesis addresses the problem of finding $a'(G)$ for a graph G which is a tensor product of either K_2, paths or cycles. We have provided optimal and sub-optimal colouring techniques for colouring the tensor product graph, whose factors are either *gap* 0 or *gap* 1 graphs. This thesis focuses on the tensor product graphs whose factors are specific *gap* 0 and *gap* 1 graphs which are edges, paths or cycles. This can later be extended to arbitrary *gap* 0 and *gap* 1 graphs.

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CHAPTER 1
Introduction

Research in the field of graph theory is evolving rapidly, with applications in new interdisciplinary science and technology. Problems from this field can be used to model and solve real problems arising in these practical domains. We illustrate the importance of graph theory in solving problems arising in practice with the following examples. There are many problems which seek to schedule,sequence, or time-table a set of events subject to given constraints and restrictions that can be modeled as graph colouring problems like vertex colouring or edge colouring.

1.1 Definitions and Notation

All graphs considered here are simple and finite. Here, $\triangle(G)$ denotes the maximum degree of a graph G. A proper colouring $\mathcal{C}$ of a graph G is the assignment of colours to the edges of the graph such that no two adjacent edges receive the same colour. The minimum number of colours used in the proper colouring of a graph G is known as the chromatic index of the graph, denoted by $\chi'(G)$.

A proper colouring $\mathcal{C}$ of the edges of a graph G is acyclic if there is no two-coloured (bichromatic) cycle in G with respect to $\mathcal{C}$. In other words, the sub-graph induced by the union of any two colour classes in $\mathcal{C}$ is a forest. The minimum number of colours required to acyclically colour the edges of graph G is known as the acyclic chromatic index of G and is denoted by $a'(G)$. The idea of acyclic colouring was mentioned first by Grünbaum[5].

The difference between acyclic chromatic index $a'(G)$ and maximum degree of the graph $\triangle(G)$ is known as the gap of the graph. It has been conjectured that the gap of any graph is at most 2.[3]

1.2 Lower Bound on Acyclic Edge Chromatic Index

1.2.1 Relation between $\chi'(G)$ and $a'(G)$

Let $\triangle(G)$ be the maximum degree of graph G. The following result is quite obvious.

$$a'(G) \geq \chi'(G) \geq \triangle(G)$$

1.2.2 Vizing Theorem

Theorem 1. *[10] For any finite, simple graph G,*

$$\triangle(G) \leq \chi'(G) \leq \triangle(G) + 1$$

We know that the value of $\chi'(G)$ is integral. So there are only two possible values of $\chi'(G) : \triangle(G)$ and $\triangle(G) + 1$.
Graphs that can be coloured with $\triangle(G)$ colours are called class one graphs. Graphs that require $\triangle(G) + 1$ colours are called class two graphs.
So for,

class 1 graphs : $a'(G) \geq \triangle(G)$
class 2 graphs : $a'(G) \geq \triangle(G) + 1$

The lower bound of Vizing theorem $\triangle(G) \leq \chi'(G)$ is fairly obvious as we need at least $\triangle$(G) colours to colour the $\triangle$(G) neighbors of the vertex.

Tight example of upper bound is $\triangle$-regular graphs on odd number of vertices. We will prove the tightness of result for Vizing theorem by method of contradiction.

Proof. Let G be a $\triangle$-regular graph on $n = 2x + 1$ vertices, for some x. On a graph of odd order, $\triangle = 2k$ for some k, otherwise the $\triangle$-regularity condition is violated. By the Handshake Lemma, G has $\triangle \cdot \frac{n}{2} = k(2x + 1)$ edges. Now, assume for contradiction that $\triangle$ colours can colour the edges of G. $\frac{k(2x+1)}{\triangle} = x + \frac{1}{2}$ (average edges per colour), so there is at least one colour that appears on $x + 1$ edges. But G only has $2x + 1$ vertices, so two of these same-coloured edges must share a vertex, and we have a contradiction. By Vizing's Theorem, the only other option for the chromatic index is $\triangle + 1$. $\square$

1.3 Upper Bound for Acyclic Chromatic Index

1.3.1 Established Upper Bound

Determining the upper bound of $a'(G)$ has been a difficult problem. Alon, McDiarmid and Reed[2] used probabilistic arguments to get an upper bound of $a'(G) \leq 64 \triangle (G)$.
This upper was then improved by Molly and Reed[7] $a'(G) \leq 16 \triangle (G)$ by making some careful observations.
Later Muthu, Narayanan and Subramanian[8] refined the results even further for graphs having girth (the length of a shortest cycle) of at least 220, which is $a'(G) \leq 4.52 \triangle (G)$

1.3.2 Conjectured Upper Bound

1.3.2.1 Conjectured result

According to the conjecture by Alon, Sudakov and Zaks[3],

$$a'(G) \leq \triangle(G) + 2$$

for every graph G.
They even showed the tight examples of upper bound.

1.3.2.2 Tightness of Conjectured Upper Bound

A tight example of this is complete graph on even vertices K_{2n}.

Proof. Let us assume that K_{2n} can be coloured with $\triangle$+1 colours. Here maximum degree $\triangle = 2n - 1$, so $\triangle + 1 = 2n$. That means we can colour the edges using $2n$ colours.
Let the colour classes be denoted by $c_1, c_2, \ldots, c_{2n}$.
We know that there can be only one colour class that is a perfect matching in an acyclic edge colouring of a graph (This is because if we had two or more perfect matchings, it will result in a bichromatic cycle, so no more than one colour class can be a perfect matching).
Without loss of generality, we assume colour class c_1 is a perfect matching, so size of c_1 would be $\frac{2n}{2} = n$ and the remaining $(2n - 1)$ colour classes can have at most $(n - 1)$ edges each.
So the total number of edges that can be coloured by these $\triangle + 1$ colours.

$= n + (2n-1) \cdot (n-1)$
$= 2n^2 - 2n + 1$
$= E(K_{2n}) - (n-1)$

This shows that $\triangle + 1$ colours are not sufficient to colour all the edges of K_{2n}. Hence, we need $\triangle + 2$ colours.These remaining $(n-1)$ edges will be coloured using the additional colour c_{2n+1}.

$\square$

CHAPTER 2

Literature Survey

This chapter includes some of the papers that were studied as a part of literature survey for the thesis. These papers show the basic nature of NP-completeness of acyclic edge colouring, derives results for graphs that are a cartesian product of factor graphs and also describes a colouring technique for sub-cubic graphs with $\triangle + 2$ colours.

2.1 Acyclic Edge Colouring for sub-cubic graphs

Sub-cubic graphs are those graphs whose maximum degree is at most 3. It has been proved that the value of acyclic chromatic number and acyclic chromatic index is at most 4 and 5 respectively. This paper presents two algorithms for proving the above facts. Let the number of vertices in the graph be n.
The below algorithm acyclically colours the edges of sub-cubic graph with 5 colours in $O(n)$ time.

Algorithm for acyclic edge colouring

"Let π be any partial colouring of G using colours in [4] and let e be any uncoloured edge.

1. If e has less than 4 coloured neighbours, then there exists a colour $\alpha \in [5]$ feasible for e. Moreover, such a colour Îś can be found in O(1) time.

2. If e has 4 coloured neighbours, then there exists a partial colouring π' of G using colours in [5] and a colour $\alpha \in [5]$ satisfying the following conditions.

    - π has the same domain as π'.
    - $\pi(x) \neq \pi'(x) \Rightarrow x \in N(v)$.
    - α is feasible for v under π'.

Moreover, both π' and α can be found in $O(1)$ time."[9]

2.2 NP-completeness for determining acyclic chromatic index

It has been proved that determining the value of $a'(G)$ even for a cubic graph ($\triangle(G) = 3$) is an NP-complete problem[4]. This is done by reducing it from another known NP-complete problem of proper edge colouring.
Let G be any cubic graph. We wish to find its acyclic chromatic index. Let H be the graph that is formed from G by replacing every edge in G with the handle that is graph F as shown in 2.1.

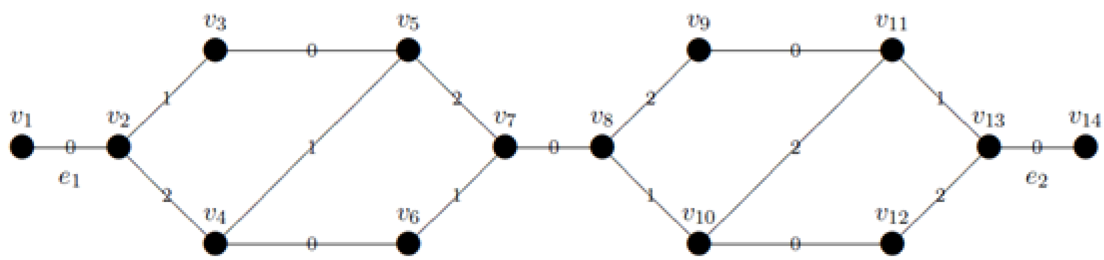

Figure 2.1: Graph F with acyclic colouring f

Let F be the graph shown above, then

1. The edges of F can be coloured acyclically using 3 colours, and there is no bichromatic path that connects v_1 and v_{14}.

2. The edges e_1 and e_2 are coloured with the same colour in any acyclic edge colouring that colours F with 3 colours.

A cubic graph G is transformed into graph H by replacing every edge in graph G by the above handle. So the proper edge colouring of G will be same as an acyclic edge colouring of H.
We can say that $a'(G)$ $\leq$3 if and only if $\chi'(G)$ $\leq$3. So we can colour the graph using 4 colours and 5 colours but determining if $a'(G)$ $\leq$3 is NP-complete.

2.3 Acyclic Edge Colouring of Grid like Graphs

[8] The paper "Optimal Acyclic Edge Colouring of Grid like Graphs" determines the values of $a'(G)$ for a special class of graphs called d-dimensional partial tori,

and denoted by $\mathcal{P}_d$. Such graphs can be expressed as a cartesian product of d graphs which are paths and cycles.

Definition 1. *A d-dimensional partial torus is a connected graph G whose unique prime factorisation is of the form $G = G_1 \square \cdots \square G_d$, where $G_i \in K_2 \cup$ PATHS $\cup$ CYCLES for each $i \leq d$. We denote the class of such graphs by $\mathcal{P}_d$.*

Definition 2. *If each prime factor of a graph $G \in \mathcal{P}_d$ is a K_2, then G is the d-dimensional hypercube. This graph is denoted by ${K_2}^d$.*

Definition 3. *If each prime factor of a graph $G \in \mathcal{P}_d$ is from PATHS, then G is a d-dimensional mesh. The class of all such graphs is denoted by $\mathcal{M}_d$.*

Definition 4. *If each prime factor of a graph $G \in \mathcal{P}_d$ is from CYCLES, then G is a d-dimensional torus. The class of all such graphs is denoted by $\mathcal{T}_d$.*

Results on Cartesian Product

"Let G be a simple graph with $a'(G) = \eta$. Then,

1. $a'(G \square P_2) \leq \eta + 1$, if $\eta \geq 2$.

2. $a'(G \square P_l) \leq \eta + 2$, if $\eta \geq 2$ and $l \geq 3$.

3. $a'(G \square C_l) \leq \eta + 2$, if $\eta \geq 2$ and $l \geq 3$

The following is true for each $d \geq 1$.

1. $a'({K_2}^d) = \triangle({K_2}^d) + 1 = d + 1$, if $d \geq 2$; $a'({K_2}^d) = 1$.

2. $a'(G) = \triangle(G) = 2d$ for each $G \in \mathcal{M}_d$.

3. $a'(G) = \triangle(G) + 1 = 2d + 1$ for each $G \in \mathcal{T}_d$.

4. Let $G \in \mathcal{P}_d$ be any graph. Let e (respectively p and c) denote the number of prime factors of G which are K_2s (respectively from PATHS and CYCLES). Then,

   (a) $a'(G) = \triangle(G) + 1 = e + 2c + 1$ if $p = 0$.

   (b) $a'(G) = \triangle(G) = e + 2p + 2c$ if either $p \geq 2$, or $p = 1$ and $e \geq 1$.

   (c) $a'(G) = \triangle(G) = 2 + 2c$ if $p = 1, e = 0$ and if at least one prime factor of G is an even cycle.

   (d) $a'(G) \in \{\triangle = 2 + 2c, \triangle + 1 = 2 + 2c + 1\}$ if $p = 0, e = 0$ and if all prime factors of G (except one path) are odd cycles."[8]

CHAPTER 3

Tensor Product

Definition 5. *Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the tensor product of G_1 and G_2, represented by $G_1 \times G_2$, is defined to be the graph $G = (V, E)$ where $V = V_1 \times V_2$ and the edge set E contains the edge joining (u_1, u_2) and (v_1, v_2) if and only if $(u_1, v_1) \in E_1$ and $(u_2, v_2) \in E_2$.*

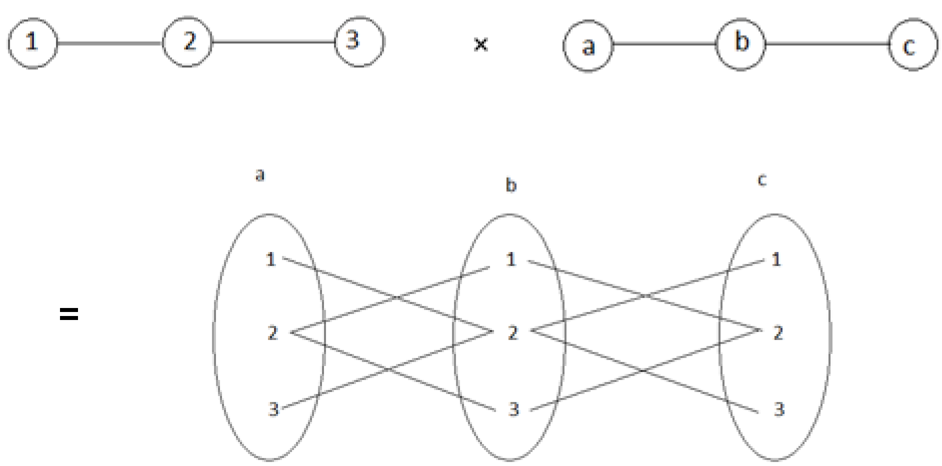

Figure 3.1: Tensor product $P_3 \times P_3$

3.1 Theorems on Tensor Product

Theorem 2. *The maximum degree of a tensor product graph is the product of individual maximum degrees. That is,*

$$\triangle(G \times H) = \triangle(G) \cdot \triangle(H)$$

Proof. Let $u \in V(G)$ be a vertex having maximum degree $\triangle(G)$ in G and $v \in V(H)$ be a vertex having maximum degree $\triangle(H)$ in H. Here, the vertex u and v will contribute towards the maximum degree of $G \times H$. The vertex (u, v) will be the vertex having maximum degree in tensor product graph $G \times H$. Vertex u is adjacent to $\triangle(G)$ vertices and vertex v is adjacent to $\triangle(H)$ vertices, hence the vertex (u, v) will be adjacent to all $\triangle(G) \cdot \triangle(H)$ possible combinations (since vertex set $V(G \times H)$ is a cartesian product of $V(G)$ and $V(H)$). Thus, maximum degree of $G \times H$ will be $\triangle(G \times H) = \triangle(G) \cdot \triangle(H)$. $\square$

Theorem 3. *The tensor product of two bipartite graphs results in a disconnected graph having exactly two components.*

Proof. The tensor product graph can be viewed as a collection of vectors in 2 dimensional space. For an edge to be present between a pair of vertices (a, b) and (c, d), there must be a change in both the co-ordinates i.e. a must migrate to b and c to d so, $a \neq b$ and $c \neq d$. So only those vertices are reachable from a given vertex (a, b) that have walks of same parity (even or odd) from both the co-ordinates. In other words, the parity of walk from a to c and b to d must be same (either odd or even). All those vertices that are reachable from a vertex form a single component. Now consider the case where we have two bipartite graphs G_1 and G_2. The partite sets of graph G_1 are labeled as L_1 and R_1 while the partite sets of graph G_2 are labeled as L_2 and R_2. So all the vertices of the tensor product graph fall into one of the following categories : (L_1, L_2) , (L_1, R_2) , (R_1, L_2) or (R_1, R_2). So an edge is possible only between the vertices of types (L_1, L_2) and (R_1, R_2) or vertices of types (L_1, R_2) and $(R_1, L_)$. So all vertices belonging to (L_1,L_2) and (R_1,R_2) belong to one component and the remaining two in another component. So we have two components but they are not necessarily isomorphic. $\square$

Theorem 4. *The acyclic chromatic index of the tensor product of two graphs is at most the product of their individual acyclic chromatic indices.*

$$a'(G \times H) \leq a'(G) \cdot a'(H)$$

Proof. The tensor product of graph G with graph H results in generation of $|V(H)|$ copies of the vertex set of G. We know that $a'(G \times K_2) \leq a'(G)$. So the tensor product of G with K_2 can be coloured using $a'(G)$ colours.

Now graph H can be viewed as a union of edges (K_2's). We add one edge e at a time and do $G \times e$. If the original edge e was coloured using colour i, then G×e would be coloured using colours from the set $\{(i-1) \cdot a'(G) + 1, (i-1) \cdot a'(G) + 2, \ldots, i \cdot a'(G)\}$.

For example, if e was coloured using colour 2, then $G \times e$ would be coloured using colours from the set $\{1 \cdot a'(G) + 1, 1 \cdot a'(G) + 2, \ldots, 2 \cdot a'(G)\}$ and if e was coloured using $a'(H)$, then $G \times e$ would be coloured using a colour from the set

$\{(a'(H)-1)\cdot a'(G)+1,(a'(H)-1)\cdot a'(G)+2,\ldots,a'(H)\cdot a'(G)\}$. Except for the fact that these different subgraphs are coloured from disjoint sets of colours, the colourings are all isomorphic to each other.
The graph $G \times H$ is coloured by introducing a new edge from H at a time and colouring in a similar fashion.
The graph $G \times H$ can be coloured using $\{1,2,3,\ldots,a'(G),a'(G)+1,\ldots,a'(G)\cdot a'(H)\}$. Thus, $a'(G\times H) \leq a'(G)\cdot a'(H)$. □

Theorem 5. *The acyclic chromatic index of a disconnected graph is the maximum of the acyclic chromatic indices of its components.*

Proof. A disconnected graph consists of two or more components. Two important properties of acyclic edge colouring is ensuring properness and absence of any bichromatic cycle. Since the components are disconnected from each other, no problem of bichromatic cycle or properness will arise between any two components. So all the components are coloured individually and the maximum acyclic chromatic index of all the components will be the acyclic chromatic index of the graph. □

Theorem 6. *The tensor product of a series of graphs is bipartite if and only if at least one graph in the series is bipartite.*

Proof. We will prove the above theorem for tensor product of two graphs, which can be extended to a series of graphs. We use the method of contradiction for the proof. Let G and H be non-bipartite graphs while $G \times H$ is a bipartite graph. This means that both G and H contains an odd cycles. We take an odd length cycle of length x from G and y from H so the tensor product of these cycles in $G \times H$ will have a cycle of length $lcm(x, y)$ which will be odd (since x and y are odd). But $G \times H$ cannot have odd cycle since we have assumed it be bipartite. This means our assumption is wrong.
Hence, at least one factor graph must be bipartite to make the resultant tensor product graph bipartite. □

Theorem 7. *The tensor product of two disconnected graphs is disconnected.*

Proof. Let G and H be the two graphs. Since G is disconnected, it will have at least two components and at least one vertex in each of the components. Let $u, v \in V(G)$ be thetwo vertices that belong to different components in G. This means that there is no u, v path.
Similarly, let $x, y \in V(H)$ be the two vertices that belong to different components in H. This means that there is no x, y path. Now consider the two vertices

$(u, x), (v, y) \in V(G \times H)$. These two vertices will be in one component if and only if there exists a u, v path and x, y path. However, these two paths are absent as discussed above. Hence there is no path to reach vertex (u, x) from (v, y). So these vertices will be in different components in the tensor product graph. This implies that the graph G×H is disconnected. □

Corollary 1. *The tensor product of two connected graphs is connected if and only if at least one of them is non-bipartite.*

Proof. As we have seen in Theorem 3, the tensor product of two bipartite graphs results in a disconnected graph, so it follows the tensor product graph will remain connected if and only if at most one factor graph is bipartite. In other words, at least one factor graph must be non-bipartite. □

Corollary 2. *The tensor product of three or more graphs is connected if and only if all of them are connected and at most one of them is bipartite.*

Proof. This is just an extension of corollary 1. As already seen in Theorem 3, the tensor product graph is connected if and only if at most one factor graph is bipartite. □

3.2 Previous work and The Defect

3.2.1 Acyclic Edge Colouring of Tensor Graph Product

The following are the results for tensor graph product as mentioned in "Acyclic colouring of Graphs"[1]:

Let G be a graph such that $a'(G) = \eta$.

- $a'(G \times P_2) \leq \eta$.
- $a'(G \times P_l) \leq \eta + \triangle(G)$, if $\eta \geq 2$ and $l \geq 3$.
- $a'(G \times C_l) \leq \eta + \triangle(G)$, if $\eta \geq 2$ and l is even.
- $a'(G \times C_l) \leq \eta + \triangle(G) + 1$, if $\eta \geq 2$ and l is odd.

The suggested colouring technique for tensor product graph of G and P_l (path on l vertices) is as follows :

" Case 1: We define the tensor product of G with K_2. For this case we define the colouring to be as follows: if an edge (u, v) was coloured using i in the original graph, then the corresponding edges i.e. $((u, x1), (v, x2))$ and $((v, x1), (u, x2))$ in

the tensor product will also be coloured using the colour i. The resultant colouring is acyclic and proper since existence of a bichromatic cycle will mean existence of a bichromatic cycle in the initial colouring of the graph G which contradicts our assumption that the initial colouring of G was acyclic.
Case 2 : We define the tensor product of graph G with P_l (path with l vertices). Here the degree of the graph has increased by $\triangle$(G) and hence we will increase the number of colours used by k and then eliminate the $2k^{th}$ colour class. The resultant colouring uses colours from 1 to $2k-1$ leaving out the $2k^{th}$ colour class, which is consistent with the conjecture. The colouring of first half of the graph, represented by C_1, is according to the algorithm given in case 1. The second half of the graph, represented by C_2, is done as follows: if an edge in C_1 is coloured with colour i then the corresponding edge in C_2 is coloured with colour $k+i$, in other words if the edge $((u, x1), (v, x2))$ is coloured i then the edge joining $((u, x2), (v, x3))$ will be coloured $k+i$. Since in any colour group we are initially using k colours which is one more than the degree of the graph, hence every vertex in x_2 has at least one colour of the colour group C_1 missing. To eliminate the $2k^{th}$ colour class we simply replace it with a missing colour of C_1 at the respective vertex of x_2. In the resultant colouring, bichromatic cycles are not possible within C_1 or C_2 as shown before. Any bichromatic cycle that exists must pass through both colour groups and include edges from both colour groups (C_1 and C_2) but such a cycle is also not possible since existence of such a cycle would require $x \in C_1$ and $y \in C_2$, which is not possible given the colouring of C_1. Having shown the colouring of $G \times P_3$, we now extend the result for $G \times P_l$. The colouring of such a graph is a direct extension of the above result. We simply alternate the colouring C_1 and C_2 defined above. The resultant colouring is proper and acyclic."[1]

3.2.2 The Defect

The above mentioned technique doesn't work for tensor product of graphs with paths having length greater than 3. The described method works for colouring of $G \times P_3$ and also in special case for $G \times P_4$ (by replicating the technique on both ends). However, this technique doesn't work in any case for tensor product of graph G with $P_n (n > 4)$.

Consider the example of $G \times P_5$:
(Here we have taken G as the simplest *gap* 1 graph which is C_3)

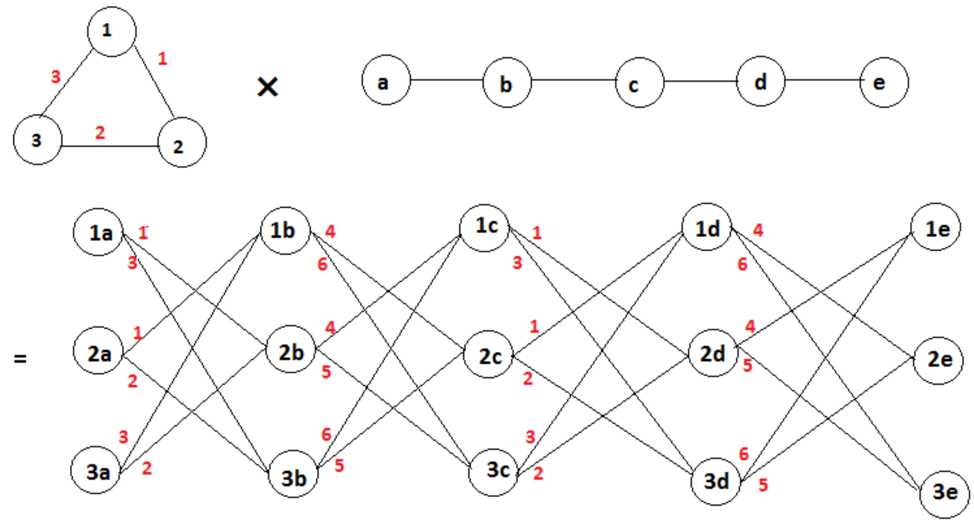

Figure 3.2: Previous Colouring scheme for $C_3 \times P_5$

This graph has been coloured as per the colouring scheme mentioned in the paper. For colouring the cross edges between the graph copies G_a and G_b, the colour i has been used as in the original graph (C_3) and for colouring the cross edges between the graph copies G_b and G_c, colour $k + i$ (here $k = 3$) has been used. The entire graph has been coloured as mentioned.

Now we wish to eliminate colour class 6.
Let us consider the edge (1b,3c) which is coloured 6 initially. For eliminating colour 6 from this edge, the edge needs to be recoloured with the minimum common colour that is free at both the endpoints so as to ensure properness.
Here, the colours available at vertex 1b are {2,5} while the colours available at vertex 3c are {1,4} so, no common colour is free at both the vertices. This means that edge (1b,3c) cannot be recoloured.
Hence the above mentioned colouring technique fails to work for tensor product of graphs with paths having length greater than 3. So obviously, this method doesn't work for higher order paths and cycles.
The following chapters provide alternative colouring techniques to resolve this issue.
However, the above mentioned technique works for tensor product of graphs with P_3, star graph and double star graph. In other words, this technique works for tensor product graphs whose one factor is a tree of maximum diameter 3.

CHAPTER 4

Tensor Product of a Graph with K_2

This chapter gives the upper bound on $a'(G)$ on tensor product of graph G with K_2. The graph could be a gap 0, gap 1 or gap 2 graph.

Given a graph G=(V,E) where $|V| = n$ and $|E| = m$.
Tensor product of this graph G with K_2 (or P_2) results in a graph $G' = (V', E')$ where $|V'| = 2n$ and $|E| = 2m$. Here the internal edges in the two copies of G disappear and the cross edges between the two copies of G appear.
The tensor product graph (G$\times K_2$) may exhibit one of the two cases, depending on graph G :

1. Preservation of gap
2. Reduction of gap

4.1 Preservation of gap

The gap of the graph is always preserved in case of tensor product with K_2.
This is because we adopt the same colouring as in the original graph. If the edge (u, v) in the original graph is coloured using colour i, then in the tensor product graph , the cross edges $((u, a), (v, b))$ and $((u, b), (v, a))$ will each be coloured using the colour i.

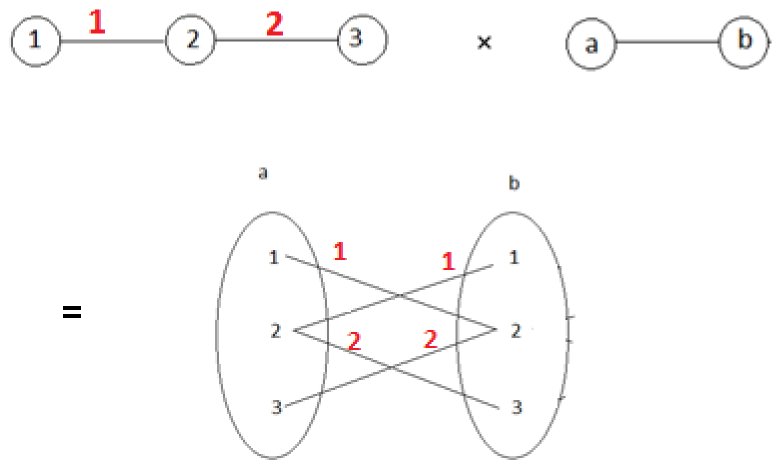

Figure 4.1: Preservation on gap on G$\times K_2$

Since the same colouring scheme is adopted, the gap of the graph is preserved.

4.2 Reduction of gap

A reduction in the gap of the graph is observed on tensor product of a special class of gap 2 graph with K_2.
This special class of graph is nothing but complete graph on n vertices, K_n where $n+1$ is prime.

$$K_6 \times K_2 \rightarrow K_{6,6} - M_6$$

Here K_6 is of the form K_{2n} and as we have already seen in the tight example of conjecture, K_{2n} is a gap 2 graph. Since the maximum degree of K_6 is 5, it can be acyclically coloured using 7 colours. A colouring of K_6 with 7 colours already exists where the edge (i, j) is coloured using the colour $(i + j) \mod 7$.
On tensor product of K_6 with K_2, we get a tensor product graph which is isomorphic to a graph which is $K_{6,6}$ minus a perfect matching on 6 vertices, M_6. See Figure 4.2. Here the edges of perfect matching M_6 that are to be eliminated is shown in red colour.
We know that $K_{6,6}$ can be coloured using 7 colours (same technique of $(i + j)$ mod 7 described above is used). $K_{6,6}$ has 36 edges and 7 colour classes so we will get 6 colour class of size 5 each and one colour class of size 6 (since we cannot have two colour classes that are perfect matchings in an acyclic edge colouring). The 6 edges that are to be removed form a perfect matching and so belong to the single colour class of size 6 (Here it is colour class 7). We eliminate this colour class

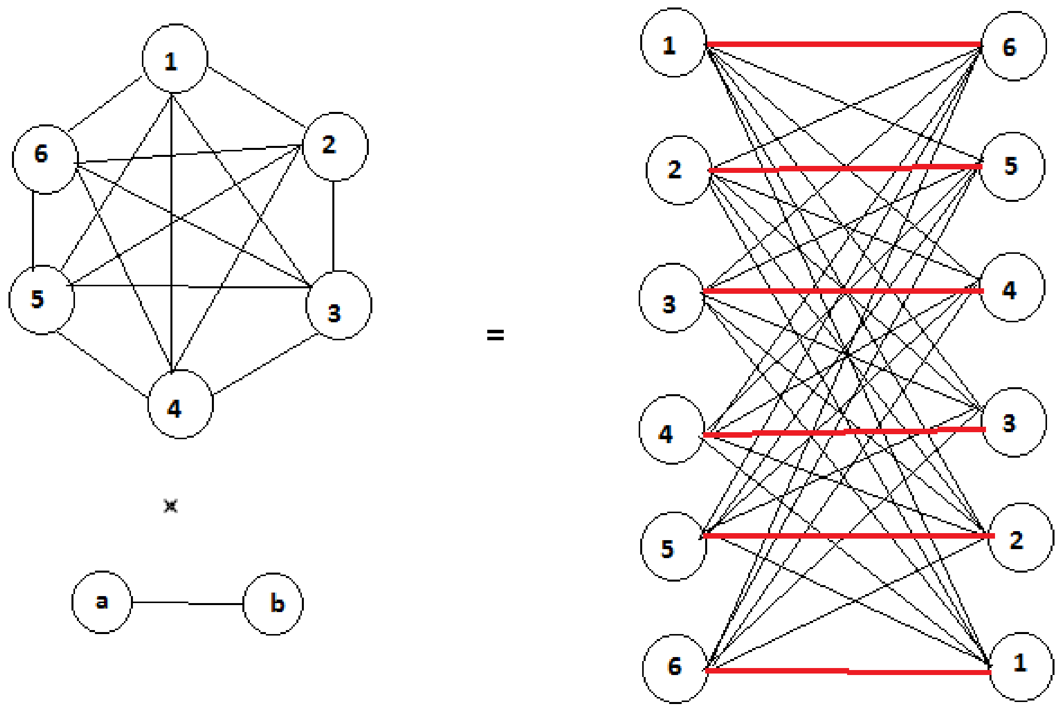

Figure 4.2: Tensor product $K_6 \times K_2$

and we are left with a graph which is the resultant tensor product graph which is coloured using 6 colours.
Here $\triangle(K_6 \times K_2)$ = 5 and $a'(K_6 \times K_2)$=6 so, it is a gap 1 graph.
Hence tensor product of K_{2n} (where $2n + 1$ is prime) with K_2 results in reduction of *gap* of the graph.

This shows that the *gap* of the graph is preserved on tensor product of graph G with K_2, and reduced in some special cases, but never increases.

CHAPTER 5
Tensor Product of a Series of Gap 0 Graphs

This chapter deals with tensor product of a series of paths (or more generally gap 0 graphs).
Gap 0 graphs are those graphs whose acyclic chromatic index is the same as the maximum degree of the graph i.e. a'(G)=$\triangle(G)$.

Theorem 8. *Tensor product of two gap 0 graphs results in a gap 0 graph.*

Proof. Let G and H be the two gap 0 graphs. So, a'(G)=$\triangle(G)$ and a'(H)=$\triangle(H)$.
We know that $a'(\mathrm{G}\times\mathrm{H}) \geq \triangle(\mathrm{G}\times\mathrm{H})$
But according to Theorem 2, $\triangle(\mathrm{G}\times\mathrm{H}) = \triangle(\mathrm{G})\cdot\triangle(\mathrm{H})$
$\therefore a'(\mathrm{G}\times\mathrm{H}) \geq \triangle(\mathrm{G})\cdot\triangle(\mathrm{H})$
According to Theorem 4, $a'(\mathrm{G}\times\mathrm{H}) \leq a'(\mathrm{G})\cdot a'(\mathrm{H})$. But $a'(\mathrm{G})=\triangle(\mathrm{G})$ and $a'(\mathrm{H})=\triangle(\mathrm{H})$.
So, $a'(\mathrm{G}\times\mathrm{H}) \leq \triangle(\mathrm{G})\cdot \triangle(\mathrm{H})$.
$\therefore a'(\mathrm{G}\times\mathrm{H}) = \triangle(\mathrm{G})\cdot \triangle(\mathrm{H}) = \triangle(\mathrm{G}\times\mathrm{H})$
So, G×H is a gap 0 graph. □

Corollary 3. *Tensor product of a series of gap 0 graphs $G_1 \times G_2 \times \cdots \times G_n$ results in a gap 0 graph.*

Proof. As seen in Theorem 8, tensor product of two gap 0 graphs results in a gap 0 graph. Tensor product of this resultant graph with another gap 0 graph again results in a gap 0 graph. Extending the result, we get a gap 0 graph on tensor product of a series of gap 0 graphs. □

CHAPTER 6

Optimal Colouring of Tensor Product of Paths and Cycles

This chapter focuses on finding the acyclic chromatic index for a graph G which is a tensor product of a path and a cycle.
The Figure 6.1 shows the tensor product $C_3 \times P_4$. While Figure 6.2 shows a graph which is isomorphic to original graph for $C_3 \times P_4$. This is nothing but a different drawing of the same graph using the Layering Technique. Henceforth, we will use the Layering Technique for a better visualization of the graph.

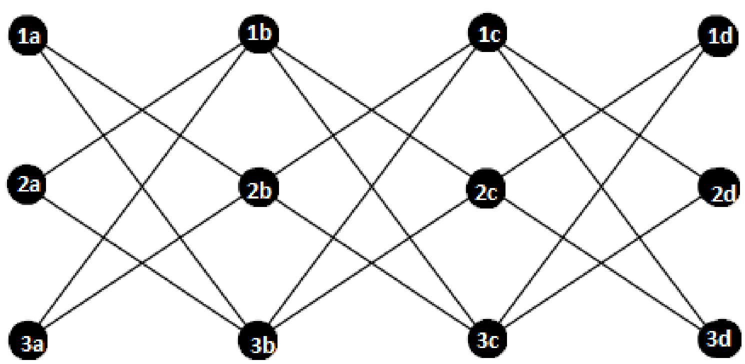

Figure 6.1: Simple drawing of Tensor product $C_3 \times P_4$

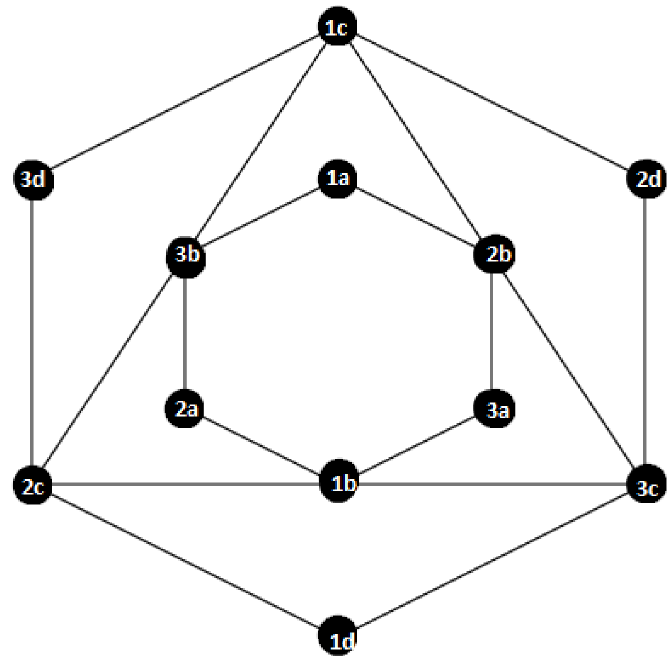

Figure 6.2: Drawing of Tensor product $C_3 \times P_4$ using Layering Technique

There are two cases for tensor product of paths and cycles :

1. Tensor product of odd length cycle and path

2. Tensor product of even length cycle and path

We have obtained optimal colourings for both the cases.

6.1 Tensor product of odd length cycle and path

We are doing tensor product of C_l (l is odd) with P_n.

Algorithm 1 Odd length cycle and Path

Input: Graph that is a tensor product of odd length cycle C_l and path P_n

Output: Acyclically edge coloured graph

1: Label the copies of graph C_l by $G_0, G_1, \ldots, G_{n-1}$.
Label the vertices in the form (j,k), where $j \in \{0,1,\ldots,l-1\}$ and $k \in \{0,1,\ldots,n-1\}$.

2: Colour the cycle between the graph copies G_{2i} and G_{2i+1} (i=0,1,2,..) by using colouring scheme C1.
Colouring C1 : A sequence of colours 1,2,1,2,. . .,1 starting from vertex $(0,2i)$ and proceeding towards $(1,2i+1)$ and a colour 3 for the last edge to prevent bichromatic cycle.

3: Colour the cycle between the graph copies G_{2i+1} and G_{2i+2} by using the colouring scheme C2.
Colouring C2 : A sequence of colours 4,3,4,3,. . .,4 starting from vertex $(0,2i+1)$ and proceeding towards $(1,2i+2)$. Assign colour 2 to the $(l-1)^{th}$ edge starting from the first edge in that cycle.

4: Repeat the above two colourings $C1$ and $C2$ followed by $C1$ with a right rotation of 1 and $C2$ with a left rotation of 1 in a circular manner till all the cycles have been coloured.

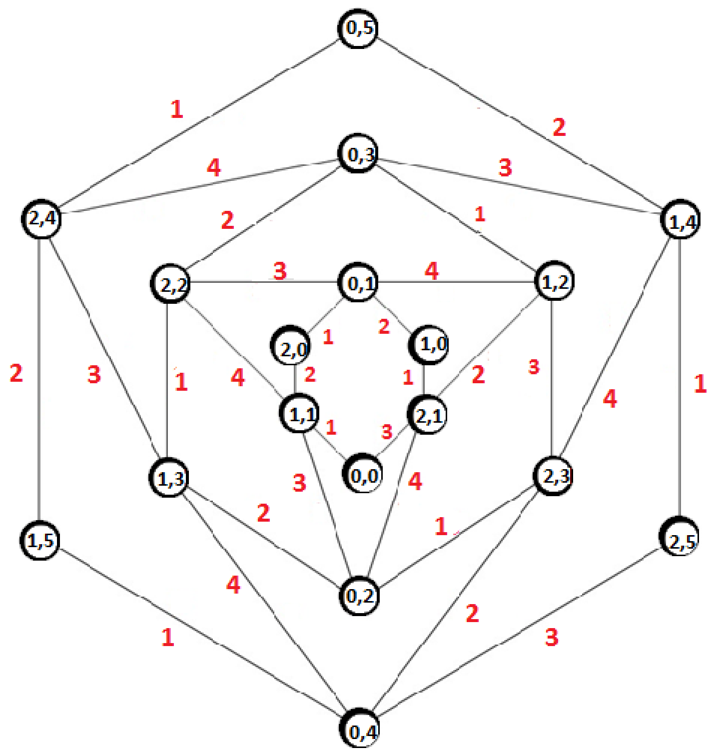

Figure 6.3: Optimal Acyclic Edge Colouring for $C_3 \times P_6$

6.2 Tensor product of even length cycle and path

We are taking tensor product C_l with P_n.

Here both the graphs are bipartite so according to Theorem 7, we get a disconnected graph with two components. But since both the graphs are highly symmetric, both the components of product graph will be isomorphic to each other so the problem of colouring $C_{2l} \times P_n$ reduces to colouring two copies of $C_l \times P_n$.

For example, the acyclic chromatic index of the tensor product of C_3 with P_n is the same as the acyclic chromatic index of the tensor product of C_6 with P_n. This is because the tensor product of C_6 with P_n is a disconnected graph consisting of two components each of which is isomorphic to C_3 tensor product P_n.

For colouring, apply the below algorithm for both the components separately.

Algorithm 2 Even length cycle and Path

Input: Graph that is a tensor product of even length cycle C_l and path P_n

Output: Acyclically edge coloured graph

1: Label the copies of graph C_l with G_0, G_1,...,G_{n-1}.
Label the vertices in the form (j, k) where $j \in \{0, 1, \ldots, l-1\}$ and $k \in \{0, 1, \ldots, n-1\}$

2: Colour the cycle between the graph copies G_{2i} and G_{2i+1},$(i = 0, 1, 2, ..)$ by using colouring scheme $C1$.
Colouring $C1$: A sequence of colours 1,2,1,2,...,1 starting from vertex $(0, 2i)$ and proceeding towards $(1, 2i+1)$ and a colour 3 for the last edge to prevent bichromatic cycle.

3: Colour the cycle between the graph copies G_{2i+1} and G_{2i+2} by using the colouring scheme $C2$.
Colouring $C2$: A sequence of colours 4,3,4,3,..., starting from vertex $(0, 2i+2)$ and proceeding towards $(1, 2i+i)$. Assign colour 2 to the $(l-1)^{th}$ edge starting from the first edge in that cycle.

4: Repeat the colourings $C1$ and $C2$ followed by $C1$ with a right rotation of 1 and $C2$ with a left rotation of 1 in a circular manner till all the cycles have been coloured.

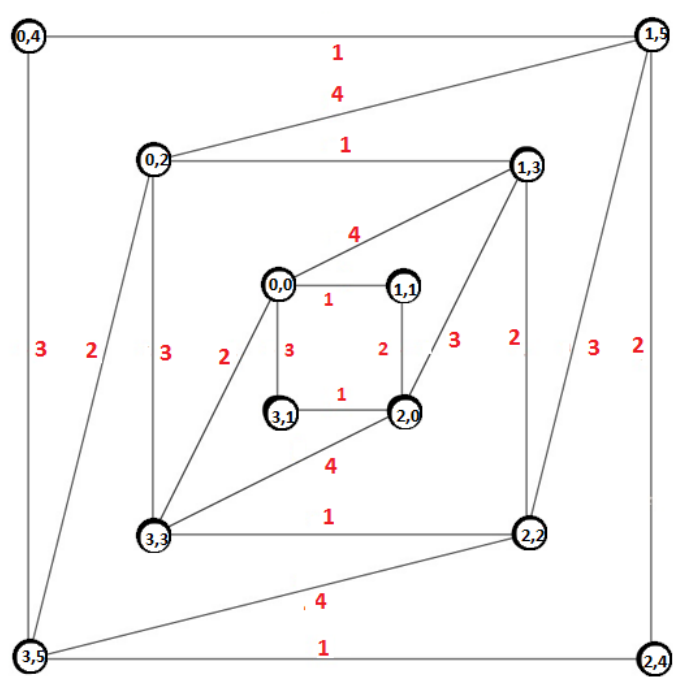

Figure 6.4: Optimal Acyclic Edge Colouring for $C_4 \times P_6$

Here the maximum degree of tensor product graph is
$\triangle(C_l \times P_n) = \triangle(C_l) \cdot \triangle(P_n) = 2{\cdot}2 = 4$

And we know that $a'(G) \geq \triangle(G)$ Therefore, $a'(C_l \times P_n) \geq 4$ and we have got an acyclic edge colouring of the graph with 4 colours and it can't get better than that.
Thus, $a'(C_l \times P_n) = 4$ or $a'(C_l \times P_n) = \triangle(C_l \times P_n)$ Also note that cycle C_l is a gap 1 graph and P_n is a gap 0 graph, while tensor product $C_l \times P_n$ is gap 0.

$$Gap1 \times Gap0 \to Gap0$$

Hence, we can say that tensor graph product experiences a reduction in the gap of the graph.

CHAPTER 7

A Sub-optimal Colouring for Tensor Product of Two Cycles

This chapter deals with acyclic edge colouring of tensor product of two cycles.

The graph C_l is a gap 1 graph, $a'(C_l)= \triangle(C_l)+1$.
Maximum degree of $C_l \times C_n$ = $\triangle(C_l) \cdot \triangle(C_n)$ = 2·2 = 4. So the tensor product of two cycles C_l and C_k will be a 4-regular graph.

Fact 1. If a graph G is regular with $\triangle(G) \geq 2$, then $a'(G) \geq \triangle(G)+1$.
This is because if $a'(G)$ is equal to $\triangle(G)$, then all $\triangle$-colours will be present at every vertex and we can get back to the same vertex following some cycle with any two colours a and b.

Since the tensor product of two cycles C_l and C_k will be a 4-regular graph, so according to Fact 1, $a'(G) \geq 5$.

Three cases are possible for tensor product of two cycles :

1. Odd length and even length cycle
2. Even length and even length cycle
3. Odd length and odd length cycle

We provide sub-optimal colouring techniques for the above cases.

7.1 Tensor product of odd length cycle and even length cycle

We are doing tensor product of an odd cycle C_l with even cycle C_n. The resultant tensor product graph is bipartite and connected. The tensor product results in creation of 2l length cycle which will be 2 mod 4 cycle.

Algorithm 3 Tensor product of odd length cycle and even length cycle

Input: Graph that is a tensor product of odd cycle C_l and even cycle C_n

Output: Acyclically edge coloured graph with 6 colours

1: Label the copies of graph C_l with G_0, G_1,...,G_{n-1}.
Label the vertices as (j,k) where $j \in \{0,1,\ldots,l-1\}$ and $k \in \{0,1,\ldots,n-1\}$.

2: Colour the cycle between the graph copies G_{2i} and G_{2i+1} $(i = 0,1,2,\ldots)$ by using colouring $C1$.
Colouring $C1$: A sequence of colours 1,2,1,2,. . .,1 starting from vertex $(0,2i)$ and proceeding towards $(1,2i+1)$. Recolour the last edge of the cycle with colour 3 to prevent bichromatic cycle.

3: Colour the cycle between the graph copies G_{2i+1} and G_{2i+2} by using the colouring $C2$.
Colouring C2 : A sequence of colours 4,5,4,5,. . .,4 starting from vertex $(0,2i+1)$ and proceeding towards $(1,2i+2)$. Recolour the last edge of the cycle with colour 6 to prevent bichromatic cycle.

4: Repeat the above two colourings $C1$ and $C2$ alternately till all the cycles except the last cycle gets coloured.

5: For colouring the last cycle i.e the cycle between graph copies G_{n-1} and G_0, use colouring $C2$ after providing a right rotation of 1.

Here there are no chances of formation of bichromatic cycle because the primary cycles of length $2l$ are always coloured using 3 colours. The last cycle between graph copies G_{n-1} and G_0 is given a right rotation of 1 so as to prevent bichromatic cycles. Moreover there will be no bichromatic cross cycles. This is because any cross cycle will include at least one edge each from both the cycles plus some extra edges, so 3 colours are used in this case also.
Figure 7.1 shows the sub-optimal colouring of tensor product graph $C_3 \times C_6$.

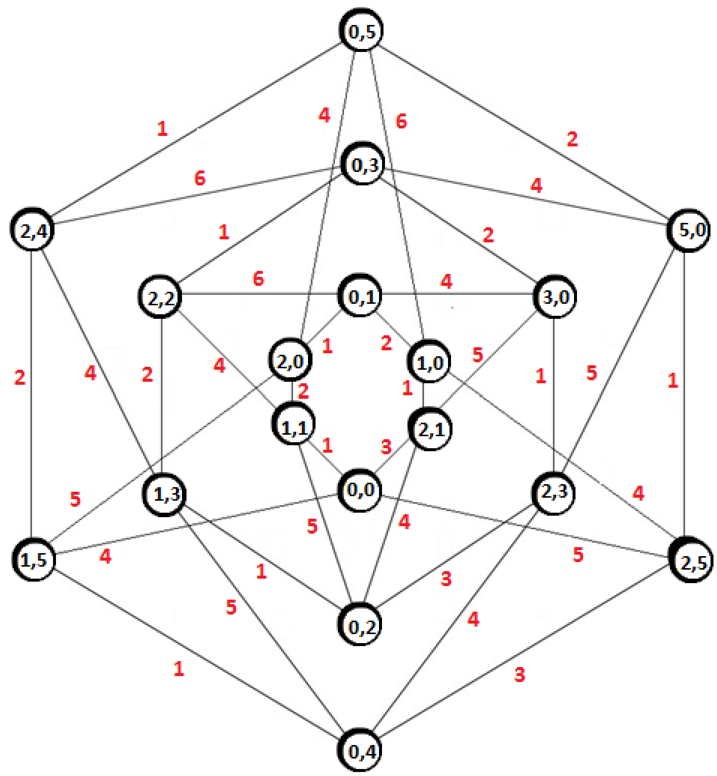

Figure 7.1: Sub-optimal Acyclic Edge Colouring of $C_3 \times C_6$

7.2 Tensor product of even length and even length cycle

Since both the graphs are bipartite, the resultant graph will be bipartite and disconnected. We apply the same colouring technique on both the components separately.

Here the colouring technique remains identical to the one for odd and even length cycle, the only difference being the vertex $(0, 2i + 2)$ from where colouring $C2$ starts.

Algorithm 4 Tensor product of even length cycle and even length cycle

Input: Graph that is a tensor product of even cycle C_l and even cycle C_n

Output: Acyclically edge coloured graph with 6 colours

1: Label the copies of graph C_l with $G_0, G_1, \ldots, G_{n-1}$.
Label the vertices as (j,k) where $j \in \{0,1,\ldots,l-1\}$ and $k \in \{0,1,\ldots,n-1\}$.

2: Colour the cycle between the graph copies G_{2i} and G_{2i+1} $(i = 0,1,2,\ldots)$ by using colouring $C1$.
Colouring $C1$: A sequence of colours 1,2,1,2,...,1 starting from vertex $(0,2i)$ and proceeding towards $(1,2i+1)$. Recolour the last edge of the cycle with colour 3 to prevent bichromatic cycle.

3: Colour the cycle between the graph copies G_{2i+1} and G_{2i+2} by using the colouring $C2$.
Colouring $C2$: A sequence of colours 4,5,4,5,...,4 starting from vertex $(0,2i+2)$ and proceeding towards $(1,2i+i)$. Recolour the last edge of the cycle with colour 6 to prevent bichromatic cycle.

4: Repeat the above two colourings $C1$ and $C2$ alternately till all the cycles except the last cycle gets coloured.

5: For colouring the last cycle i.e the cycle between graph copies G_{n-1} and G_0, use colouring $C2$ after providing a right rotation of 1.

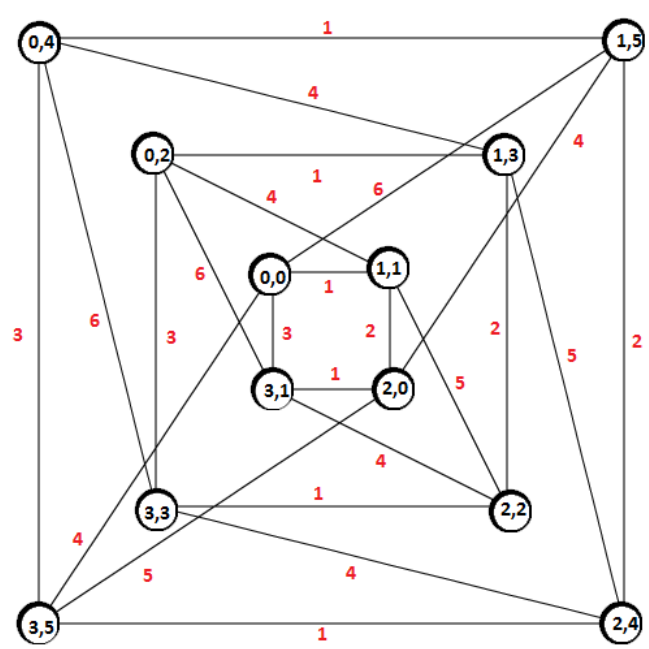

Figure 7.2: Sub-optimal Acyclic Edge Colouring of $C_4 \times C_6$

CHAPTER 8

Optimal Colouring for Tensor Product of Two Cycles

The previous chapter gave colouring schemes for sub-optimal colouring of tensor product of two cycles. This chapter improves the results for acyclic edge colouring of tensor product of two cycles by providing optimal colouring scheme.

Fact 1. If a graph G is regular with $\triangle$(G) $\geq$ 2, then $a'(G) \geq \triangle(G) + 1$.
This is because if $a'(G)$ is equal to $\triangle(G)$, then all $\triangle$-colours will be present at every vertex and we can get back to the same vertex following some cycle with any two colours a and b.

We know that the tensor product of two cycles C_l and C_k will be a 4-regular graph and so $a'(G) \geq 5$. We provide colouring scheme that colours the tensor product graph $C_l \times C_n$ with 5 colours.

The tensor product of two cycles C_l and C_k results in a graph which is isomorphic to a known class of graphs, chordal ring $C_N\langle n\rangle$ of degree 4, where $N = l * k$ and $n = \{k-1, (l-3)k + \lceil \frac{k}{2} \rceil + 2\}$. A chordal ring $C_N\langle n\rangle$ can be decomposed into two isomorphic Hamiltonian cycles[6].

We partition the graph which is tensor product of two cycles into two Hamiltonian cycles by using Algorithm 5 described below.

Algorithm 5 Decomposition of graph into Two Hamiltonian cycles

Input: Tensor product graph whose one factor is an odd cycle

Output: Two Edge-Disjoint Hamiltonian cycles H_1 and H_2

1: Label the vertices of $C_l \times C_n$ as (i, j) where $i \in \{0, 1, \ldots, l-1\}$ and $j \in \{0, 1, \ldots, n-1\}$
2: Start with $i = 0$ and $j = 0$. Visit vertex (i, j).
3: **if** j $\neq$ n-1 **then**
4:     if j is even, $i = (i+1) \bmod l$ and $j = j+1$.
5:     if j is odd, $i = (i-1) \bmod l$ and $j = j+2$.
6: **else if** $j = n-1$ **then**
7:     $i = (i+1) \bmod l$ and $j = 0$.
8: Visit vertex (i, j).
9: Repeat Steps-3 through 8 till $i = 0$ and $j = 0$.
10: Output cycle joining all the above visited vertices in order as Hamiltonian cycle H_1.
11: Connect remaining edges of graph $C_l \times C_n$ and output it as Hamiltonian cycle H_2.

Consider the example of tensor product of C_3 and C_5. The resultant tensor product graph $C_3 \times C_5$ will have 3∗5=15 vertices. So we will get two Hamiltonian cycles of length 15. The decomposition of C_3 and C_5 into Hamiltonian cycles H_1 and H_2 according to Algorithm 5 is shown below.

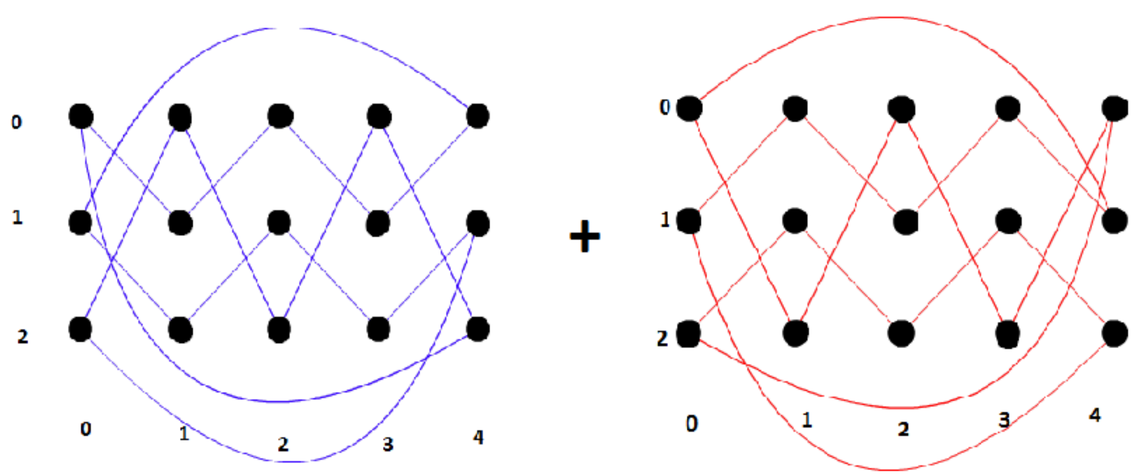

Figure 8.1: Decomposition of $C_3 \times C_5$ into 2 Hamiltonian Cycles

Algorithm 6 Tensor product $C_l \times C_n$ where n is even

Input : Graph that is tensor product of two cycles and at least one of them is an even cycle

Output : Acyclically edge coloured graph with 5 colours

1: Decompose the tensor product graph $C_l \times C_n$ into two Hamiltonian cycles H_1 and H_2 using Algorithm 5.

2: Colour Hamiltonian cycle H_1 with an alternating sequence of colours 1 and 2, starting from (0,0) and proceeding towards (1,1).
Recolour the first $\frac{n}{2}$ occurrences of colour 1 in the cycle H_1 by colour 3.

3: Colour Hamiltonian cycle H_2 with an alternating sequence of colours 4 and 5, starting from (0,0) and proceeding towards $(1, n-1)$.
Recolour the next $\frac{n}{2}$ occurrences of colour 4 after $(n+1)^{th}$ edge in the cycle H_2 by colour 3.

We consider the following three cases :

1. Odd length cycle and even length cycle
2. Even length cycle and even length cycle
3. Odd length cycle and odd length cycle

Case 1. Odd length cycle and even length cycle

We are doing tensor product $C_l \times C_n$ where l is odd and n is even. The resultant tensor product graph will be bipartite and connected.
Figure 8.2 shows the optimal colouring of $C_3 \times C_4$ after applying Algorithm 6

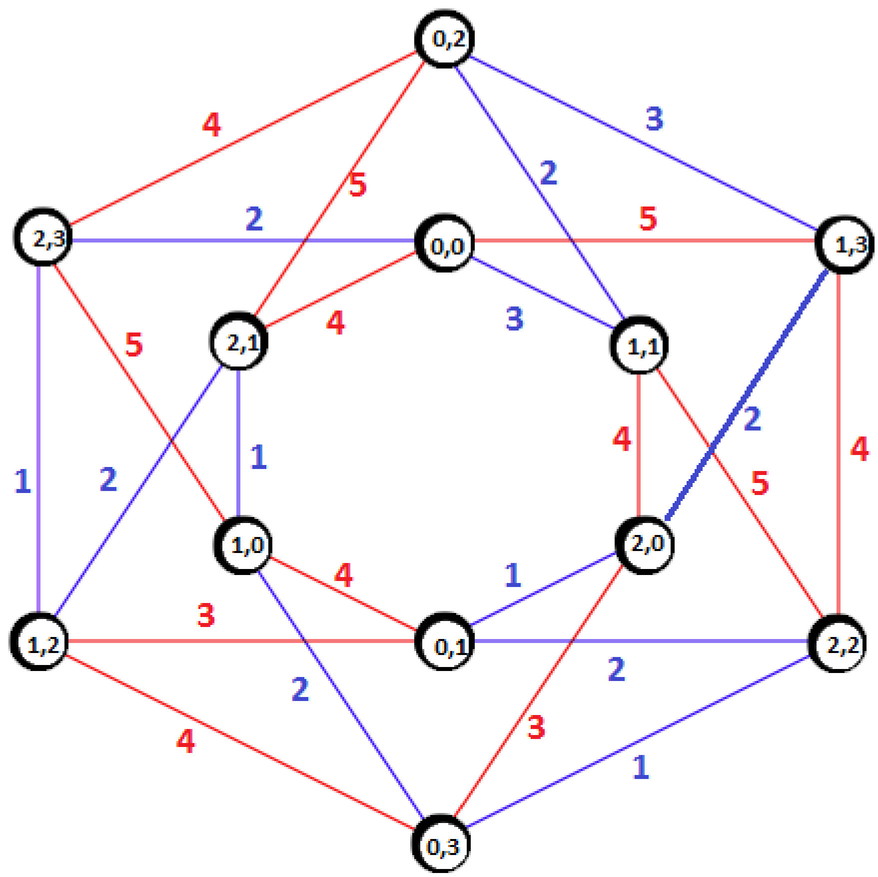

Figure 8.2: Optimal Acyclic Edge Colouring of $C_3 \times C_4$

Case 2. Even length cycle and even length cycle

We are doing tensor product $C_l \times C_n$ where both l and n are even. The resultant tensor product graph will be bipartite and disconnected.
We apply Algorithm 6 to each of the components separately. Each component gets decomposed to two Hamiltonian cycles and so entire graph gets disconnected into 4 Hamiltonian cycles (in the respective components).
We know that $C_l \times C_n$ is isomorphic to $C_n \times C_l$. So in case of tensor product of two even cycles, we will always construct the graph for $C_l \times C_n$ where l≤n.
For example, for tensor product of C_4 and C_8, we will always do $C_8 \times C_4$.
Figure 8.3 shows the optimal colouring of one component of $C_8 \times C_4$.

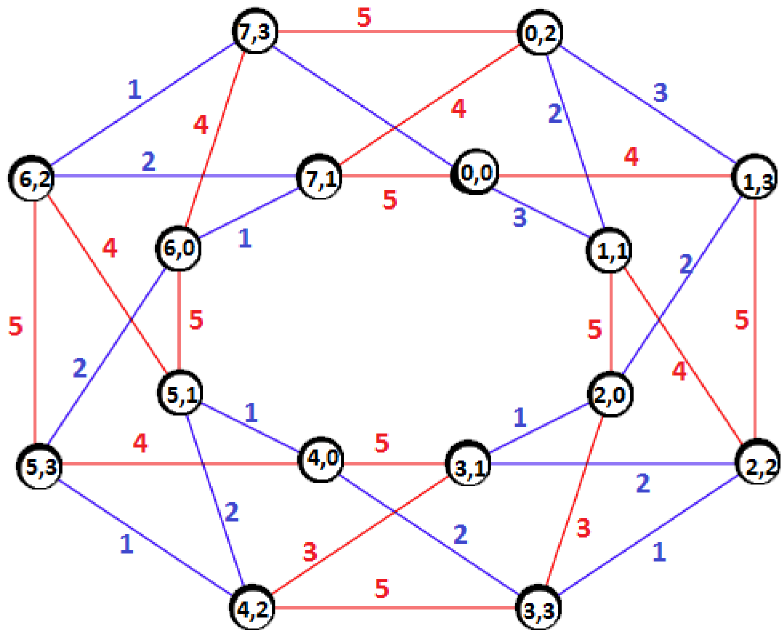

Figure 8.3: Optimal Acyclic Edge Colouring of one component of $C_8 \times C_4$

Case 3. Odd length cycle and odd length cycle

We are doing tensor product $C_l \times C_n$ where both l and n are odd. The resultant tensor product graph will be non-bipartite and connected.
Here also we do the same case like even and even cycle. For tensor product of two odd cycles C_l and C_n, we will always do $C_l \times C_n$ where l≤n.

Figure 8.4 shows the optimal colouring of $C_3 \times C_3$ using Algorithm 7.

Algorithm 7 Tensor product of two odd cycles $C_l \times C_n$

Input : Graph that is tensor product of two odd cycles C_l and C_n
Output : Acyclically edge coloured graph with 5 colours

1: Decompose the tensor product graph $C_l \times C_n$ into two Hamiltonian cycles H_1 and H_2 using Algorithm 5.
2: Colour Hamiltonian cycle H_1 with an alternating sequence of colours 1 and 2, starting from (0,0) and proceeding towards (1,1).
Recolour the first $\lfloor \frac{n}{2} \rfloor$ occurrences of colour 1 in the cycle H_1 by colour 3.
3: Colour Hamiltonian cycle H_2 with an alternating sequence of colours 4 and 5, starting from (0,0) and proceeding towards $(1, n-1)$.
Recolour the next $\lfloor \frac{n}{2} \rfloor$ occurrences of colour 4 after 2^{nd} edge in the cycle H_2 by colour 3. Also, insert colour 3 in $(n-1)^{th}$ position from the end of cycle.

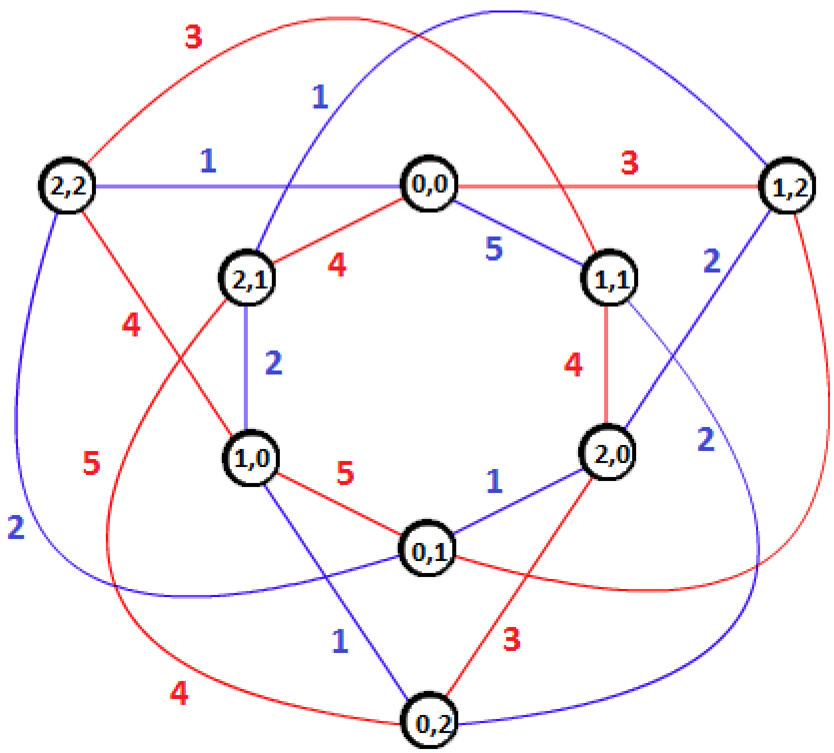

Figure 8.4: Optimal Acyclic Edge Colouring of $C_3 \times C_3$

Here the lower bound on acyclic chromatic index of tensor product of two cycles was $\triangle$(G)+1 and we have obtained a colouring with $\triangle$(G)+1 colours.
Hence a'(G) = $\triangle$(G)+1 for tensor product of two cycles. Cycle was a gap 1 graph and resultant tensor product graph is also gap 1. This shows that the gap of the graph is preserved.

Chapter 9

Conclusion

We have obtained optimal colouring techniques for graph G whose factors are either paths or cycles. We showed that the gap of the graph is preserved on tensor product of cycles (a special class of gap 1 graphs) with paths and cycles.
This study can be extended to multi dimensions by doing tensor product of a series of paths and cycles. Here we have considered product graphs which has two irreducible factors which is paths or cycles. The future scope involves the acyclic edge colouring of d-dimensional product graphs which can be decomposed into d irreducible factors which are cycles and paths.
This study can further be extended to verify the preservation of gap on tensor product of arbitrary gap 1 graphs with paths, cycles and trees.

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